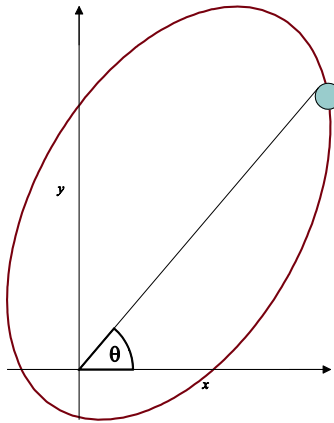


LONG QUESTIONS: MARKING SCHEME

1. A moon is orbiting a planet such that the orbit is perpendicular to the surface of the planet where an observer is standing. After some necessary scaling, suppose the orbit satisfies the following equation

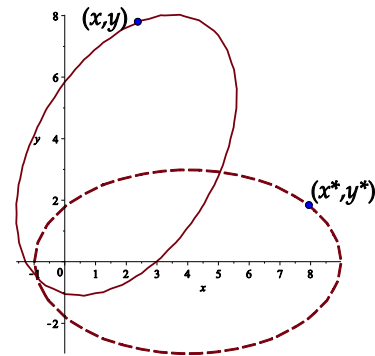
$$9\left(\frac{x}{2} + \frac{\sqrt{3}y}{2} - 4\right)^2 + 25\left(-\frac{\sqrt{3}x}{2} + \frac{y}{2}\right)^2 = 225$$

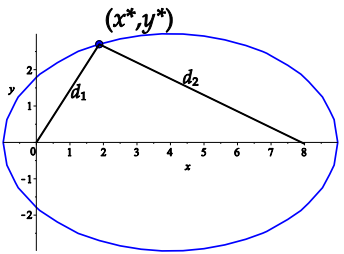
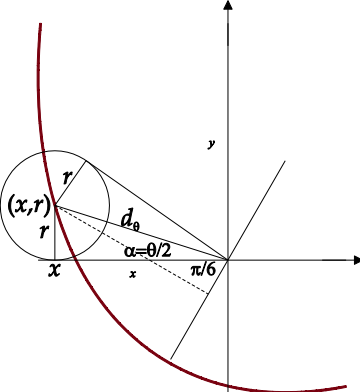
Let r be the radius of the moon. Assume that the period of rotation of the planet is much larger than the orbital period of the moon. Determine $\tan \frac{\theta}{2}$, where θ is the elevation angle when the moon looks largest to the observer.



Answer and Marking Scheme:

1	<p>Notice the standard version of the orbits</p> <p>The ellipse may be obtained from a standard ellipse</p> $\frac{(x^* - 4)^2}{25} + \frac{(y^*)^2}{9} = 1$ <p>by rotating the standard ellipse with respect to the origin, counterclockwise, (by $\pi/3$ radians). Then $c = \sqrt{a^2 - b^2} = \sqrt{16} = 4$</p>	10
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2	<p>Established relation between coordinates before and after transformation.</p> <p>For any point (x, y) on the ellipse, let (x^*, y^*) be its coordinates before the transformation. From the equation of the ellipse, we can easily see that</p> $x^* = \frac{x}{2} + \frac{\sqrt{3}y}{2}$ $y^* = -\frac{\sqrt{3}x}{2} + \frac{y}{2}$ <p>(In fact, $\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{x}{2} + \frac{\sqrt{3}y}{2} \\ -\frac{\sqrt{3}x}{2} + \frac{y}{2} \end{bmatrix}$.)</p> 	20
3	<p>Obtain expression for calculating distance from a point on an ellipse to the foci.</p> <p>Consider standard ellipse</p> $\frac{(x^* - 4)^2}{5^2} + \frac{(y^*)^2}{3^2} = 1$ <p>Choose any point (x^*, y^*) on the ellipse. Let d_1 and d_2 be the distances from any point (x^*, y^*) on the ellipse to the foci $(0,0)$ and $(8,0)$; $d_1^2 = (x^*)^2 + (y^*)^2$ and $d_2^2 = (x^* - 8)^2 + (y^*)^2$</p> <p>Thus,</p> $d_1^2 = (x^*)^2 + (y^*)^2 = (x^*)^2 + \left(9 - \frac{9(x^* - 4)^2}{5^2}\right) = \frac{(16(x^*)^2 + 72x^* + 81)}{25}$	30
4	<p>Identify and characterized the coordinates when the planet looks largest.</p> <p>The following is the position of the moon when it looks largest. At that point, its second coordinate equals r. Thus, the coordinates be (x, r).</p> 	20
5	<p>Solve equation $\frac{(16(x^*)^2 + 72x^* + 81)}{25} = x^2 + r^2$ to obtain value of x.</p> <p>Therefore</p> $d_1^2 = \frac{(16(x^*)^2 + 72x^* + 81)}{25} = x^2 + r^2$	15

	<p>where $x^* = \frac{x}{2} + \frac{r\sqrt{3}}{2}$. Therefore, we can obtain the value of x by solving the above equation for x. Substituting $x^* = \frac{x}{2} + \frac{r\sqrt{3}}{2}$ to the equation, we have</p> $25d_1^2 = 25x^2 + 25r^2 = 16\left(\frac{x}{2} + \frac{r\sqrt{3}}{2}\right)^2 + 72\left(\frac{x}{2} + \frac{r\sqrt{3}}{2}\right) + 81 =$ <p>or</p> $25x^2 + 25r^2 = 4x^2 + (8r\sqrt{3} + 36)x + (12r^2 + 36\sqrt{3}r + 81)$ $21x^2 - (8r\sqrt{3} + 36)x + (13r^2 - 36r\sqrt{3} - 81) = 0$ <p>Then use quadratic formula to obtain x in term of r. Choose smaller value of x to get larger value of $\tan \frac{\theta}{2} = \frac{r}{x}$. The smaller one is</p> $x = \frac{4r\sqrt{3} + 18}{21} - \frac{15\sqrt{-r^2 + 4r\sqrt{3} + 9}}{21}$	
6	<p>Find the expression of $\tan \frac{\theta}{2}$.</p> <p>Hence</p> $\tan \frac{\theta}{2} = \frac{r}{x} = \frac{21r}{4r\sqrt{3} + 18 - 15\sqrt{-r^2 + 4r\sqrt{3} + 9}}$	5

2. Two massive stars A and B with mass m_A and m_B are separated by a distance d . Both stars are orbiting each other with respect to their center of gravity whose orbits are circular. Suppose the stars lie on the X-Y plane (see Figure 2) and are moving under gravitational force.

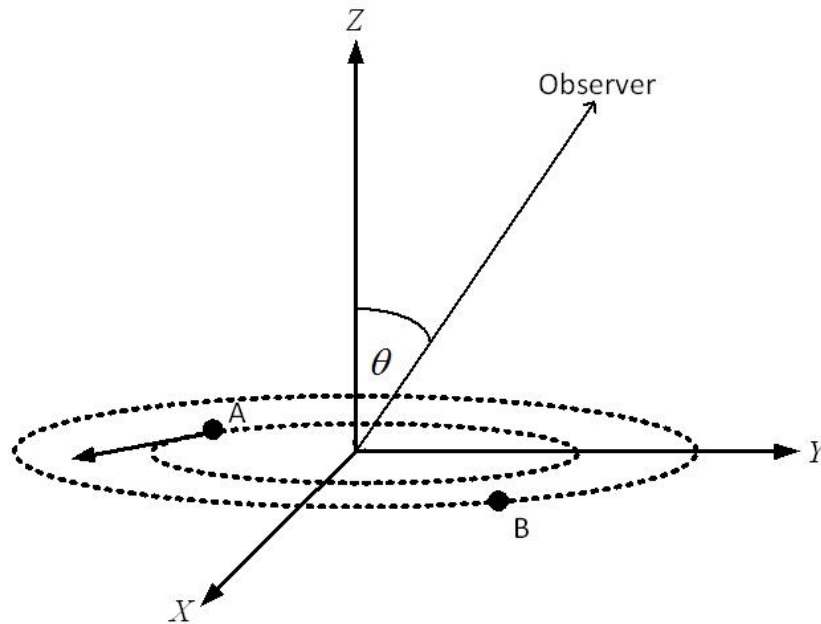


Figure 2

- a. Calculate the speed of star A and its angular velocity.

An observer lies on the Y-Z plane (see Figure 2) see the stars from the large distance with angle θ relatively to the Z-axis. He/she measure that the velocity component of A to the line of his/her sight has the form $K \cos(\omega t + \varepsilon)$, with K and ε are positive.

- b. Express the value $K^3/\omega G$ in term of m_A , m_B , and θ where G is the universal gravitational constant.

The observer can then identify that the star A has mass equal to $30M_S$ where M_S is the Sun's mass. On the other hand, he/she observe that the star B produces X-rays, so it could be a neutron star or a black hole. These situations depend on m_B : 1) If $m_B < 2M_S$, then B is a neutron star. 2) If $m_B > 2M_S$, then B is a black hole.

- c. A measurement has been done by the observer which results $\frac{K^3}{\omega G} = \frac{1}{250} M_S$. If the value of $\cos \theta$ has the same probability, then calculate the probability of B to be a black hole. (Hint: Use $\int \sin x \, dx = -\cos x + C$)

Answer and Marking Scheme:

a.	<p>The center of gravity of the stars is relatively to the star A given by</p> $r_A = \frac{m_A}{m_A + m_B} d$ <p>and since the orbit of A is a circle, then</p> $F_{AX} = \frac{G m_A m_B}{d^2} = \frac{m_A v_A^2}{r_A}$ <p>So, we get</p> $v_A = m_B \sqrt{\frac{G}{(m_A + m_B) d}}$ <p>The angular velocity of A is given by</p> $\omega = \frac{v_A}{r_A} = \sqrt{\frac{G(m_A + m_B)}{d^3}}$	30
b.	<p>In Cartesian coordinate system, the velocity of A is</p> $\vec{v}_A(t) = v_A(-\sin(\omega t + \varepsilon) \hat{i} + \cos(\omega t + \varepsilon) \hat{j})$ <p>Unit vector of the observer is</p> $\hat{r}_P = \cos \theta \hat{k} + \sin \theta \hat{j}$ <p>so the component of \vec{v}_A in the line of the observer sight is given by</p> $\vec{v}_A \cdot \hat{r}_P = v_A \sin \theta \cos(\omega t + \varepsilon)$ <p>Since the component of \vec{v}_A in the line of the observer sight is $K \cos(\omega t + \varepsilon)$, then</p> $K = v_A \sin \theta$ <p>Finally, we have</p> $\frac{K^3}{\omega G} = \frac{m_B^3}{(m_A + m_B)^2} \sin^3 \theta$ <p style="text-align: right;">(**)</p>	30
c.	From the result in b., namely eq. (**), we get	40



$$\sin^3 \theta = \frac{K^3}{\omega G} \frac{(m_A + m_B)^2}{m_B^3} < \frac{1}{250} \frac{32^2}{8} = \frac{64}{125}$$

Since $\theta \in [0, \pi]$, then $\sin \theta < 0,8$. Thus, the probability of B is a black hole is the same as the probability of $\sin \theta < 0,8$ for $\theta \in [0, \pi]$. Since $\sin^{-1} 0,8 \approx 53^\circ$ or 127° , then

$$P(\text{B: Black Hole}) = 1 - \frac{\int_{53^\circ}^{127^\circ} \sin \theta \, d\theta}{\int_{0^\circ}^{180^\circ} \sin \theta \, d\theta} = 1 - 0,6 = 0,4$$

3. Suppose a static spherical star consists of N neutral particles with radius R (see Figure 1).

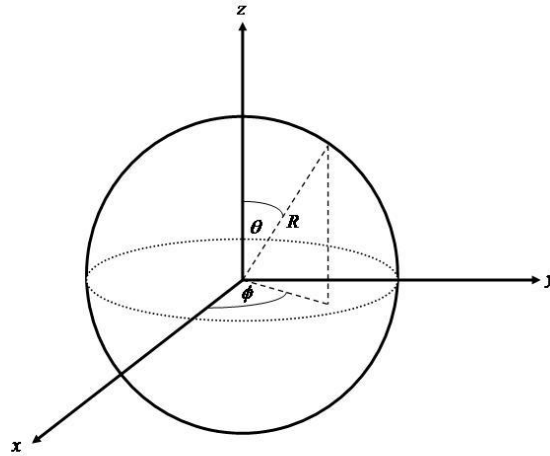


Figure 1

with $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, satisfying the following equation of states

$$P V = N k \frac{T_R - T_0}{\ln(T_R/T_0)} \quad (1)$$

where P and V are the pressure inside the star and the volume of the star, respectively, and k is Boltzmann constant. T_R and T_0 are the temperature at the surface $r = R$ and the temperature at the center $r = 0$, respectively. Assume that $T_R \leq T_0$.

- a. Simplify the stellar equation of states (1) if $\Delta T = T_R - T_0 \approx 0$ (this is called ideal star) (Hint: Use the approximation $\ln(1+x) \approx x$ for small x)

Suppose the star undergoes a quasi-static process, in which it may slightly contract or expand, such that the above stellar equation of states (1) still holds.

- b. Find the work of the star when it expands from V_1 to V_2 in isothermal process where T_R and T_0 are constants.

The star satisfies first law of thermodynamics

$$Q = \Delta M c^2 + W \quad (2)$$

where Q , M , and W are heat, mass of the star, and work respectively, while c is the light speed in the vacuum and $\Delta M \equiv M_{\text{final}} - M_{\text{initial}}$.

In the following we assume T_0 to be constant, while $T_R \equiv T$ varies.

- c. Find the heat capacity of the star at constant volume C_v in term of M and at constant pressure C_p expressed in C_v and T (Hint: Use the approximation $(1+x)^n \approx 1+nx$ for small x)

Assuming that C_v is constant and the gas undergoes the isobar process so the star produces the heat and radiates it outside to the space.

- d. Find the heat produced by the isobar process if the initial temperature and the final temperature are T_i dan T_f , respectively.
- e. Suppose there is an observer far away from the star. Related to point d., estimate the distance of the observer if the observer has 0.1% error in measuring the effective temperature around the star.

Now we take an example that the star to be the Sun of the mass M_\odot , its radius R_\odot , its luminosity (radiation energy emitted per unit time) L_\odot , and the Earth-Sun distance, d_\odot .

- f. If the sunlight were monochromatic with frequency 5×10^{14} Hz, estimate the number of photons radiated by the Sun per second.
- g. Calculate the heat capacity C_v of the Sun assuming its surface temperature runs from 5500 K until 6000 K in this period.

Answer and Marking Scheme:

a.	<p>Defining $\Delta T = T_R - T_0$ and $\Delta T \approx 0$, we have</p> $P V = N k \frac{\Delta T}{\ln(1 + \Delta T/T_0)}$ <p style="text-align: right;">(3)</p> <p>using $\ln(1 + \Delta T/T_0) \approx \Delta T/T_0$, we then obtain</p> $P V = N k T_0$ <p style="text-align: right;">(4)</p>	10
b.	<p>If the gas undergoes isothermal process where T_R and T_0 are constant, then the work has the form</p> $W = \int_{V_1}^{V_2} P dV = N k \frac{T_R - T_0}{\ln(T_R/T_0)} \ln\left(\frac{V_2}{V_1}\right)$	15



c.	<p>The internal energy of the star is $U = Mc^2$ ($U(T) = M(T)c^2$ for ideal star). Thus, the constant volume heat capacity of the star has the form</p> $C_v = \left(\frac{\Delta Q}{\Delta T} \right)_v = \left(\frac{\Delta M}{\Delta T} \right)_v c^2$ <p>for small ΔT. Then, using first law of thermodynamics, the constant pressure heat capacity of the star is</p> $C_p = \left(\frac{\Delta Q}{\Delta T} \right)_p = \left(\frac{\Delta M}{\Delta T} \right)_v c^2 + P \frac{\Delta V}{\Delta T} = C_v + P \frac{\Delta V}{\Delta T}$ <p>for small ΔT. Defining $\Delta T = T_2 - T_1$, then</p> $P \Delta V = N k \left(\frac{T_1 - T_0 + \Delta T}{\ln((T_1 + \Delta T)/T_0)} - \frac{T_1 - T_0}{\ln(T_1/T_0)} \right)$ <p>Using the approximation</p> $\ln((T_1 + \Delta T)/T_0) \approx \ln\left(\frac{T_1}{T_0}\right) + \frac{\Delta T}{T_1}$ $\left(1 + \frac{\Delta T}{T_1 \ln(T_1/T_0)} \right)^{-1} \approx 1 - \frac{\Delta T}{T_1 \ln(T_1/T_0)}$ <p>then we have</p> $P \frac{\Delta V}{\Delta T} = \frac{N k}{\ln(T/T_0)} \left(1 - \frac{(T - T_0)/T}{\ln(T/T_0)} \right)$ <p>where $T_1 \equiv T$. Finally, we obtain</p> $C_p = \left(\frac{\Delta Q}{\Delta T} \right)_p = \left(\frac{\Delta M}{\Delta T} \right)_v c^2 + P \frac{\Delta V}{\Delta T} = C_v + \frac{N k}{\ln(T/T_0)} \left(1 - \frac{(T - T_0)/T}{\ln(T/T_0)} \right)$	25
d.	<p>Since C_v is constant, the heat produced by the star is given by</p> $Q_H = C_v (T_f - T_i) + P \Delta V$ $= C_v (T_i - T_f) + N k \left(\frac{T_f - T_0}{\ln(T_f/T_0)} - \frac{T_i - T_0}{\ln(T_i/T_0)} \right)$	20
e.	<p>The error 0,1% , namely</p> $\frac{T_f - T_{eff}}{T_f} = \frac{1}{1000}$ <p>then $T_{eff} = \frac{999}{1000} T_f$. From Stefan law of black body radiation</p>	15

	$\frac{Q_H}{4\pi r_0^2} = \sigma T_{eff}^4 = \sigma \left(\frac{999}{1000}\right)^4 T_f^4$ <p>where r_0 is the observer's distance which is given by</p> $r_0 = \left(\frac{Q_H}{4\pi\sigma}\right)^{1/2} \left(\frac{1000}{999}\right)^2 T_f^{-2}$	
f.	<p>Energy per second radiated by the Sun</p> $L_{\odot} = N h \nu$ <p>where N is the number of photon. Thus</p> $N = \frac{L_{\odot}}{h \nu} = \frac{3.96 \times 10^{26}}{6.6261 \times 10^{-34} \times 5 \times 10^{14}} = 1.195 \times 10^{45} \text{ photons}$	7
g.	<p>Energy per second radiated by the Sun is proportional to mass defect of the Sun</p> $L_{\odot} = \Delta M c^2$ <p>Thus,</p> $C_v = \frac{\Delta M c^2}{\Delta T} = \frac{L_{\odot}}{\Delta T} = \frac{3.96 \times 10^{26}}{6000 - 5500} \text{ J/K} = 7.92 \times 10^{23} \text{ J/K}$	8